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B.M.S COLLEGE FOR WOMEN, AUTONOMOUS

BENGALURU – 560004

SEMESTER END EXAMINATION –APRIL / MAY- 2023

M.Sc. Mathematics – III Semester

LINEAR ALGEBRA

Course Code MM301T **QP Code: 13001**

Duration : 3 Hours **Maximum Marks: 70**

Instructions: *1)* ***All*** *questions carry* ***equal*** *marks.*
2) Answer ***any five*** *full questions.*

1. a) Define an algebra. If A is an algebra with unit element over F then prove that A is isomorphic to a subalgebra of $A_F(V)$ for some vector space V over F.
   b) Define regular and singular transformation with examples for each. If V V is a finite dimensional vector space over F then prove that for any $T \in A_F(V)$ is singular if and only if there exists a vector $v \in V$ with $v \neq 0$ such that $vT = 0$.
   c) If V is an n-dimensional vector space over F, then prove that for a given $T \in A(V)$ there exists a nontrivial polynomial $q(x) \in F[x]$ of degree atmost n^2 such that $q(T) = 0$.
   (5+5+4)

2. a) Define minimum polynomial of a linear transformation. Let $S, T \in A_F(V)$. If S is regular then prove that T and STS^{-1} have the same minimal polynomial.
   b) Define characteristic root of a linear transformation. If λ is a characteristic root of $T \in A_F(V)$ then prove that λ is a root of the minimal polynomial of T. Further prove that T has only a finite number of characteristic roots in F.
   c) If V is an n-dimensional vector space over a field F and if $T \in A_F(V)$ has the matrix $M_1(T)$ in the basis $\{v_1, v_2, \cdots, v_n\}$ and the matrix $M_2(T)$ in the basis $\{w_1, w_2, \cdots, w_n\}$ of V, then prove that there exists a matrix $C \in F_n$ such that $M_2(T) = CM_1(T)C^{-1}$.
   (4+5+5)

3. a) Define composition of two linear transformation. Prove that the product of two linear transformation is a linear transformation.
   b) Define a linear functional and dual of a basis. Prove that the double dual of V is isomorphic to V.
   c) Let $b_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, b_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, c_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ and $c_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. Consider the bases $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$. Find the change of coordinate matrix from B to C and from C to B.
   (5+5+4)

4. a) If $W \subset V$ is an invariant subspace under T, then prove that T induces a linear transformation $\bar{T}$ on $\bar{V}$. If T satisfies a polynomial $q(x) \in F[x]$, then prove that $\bar{T}$ also satisfies $q(x)$. Further, if $p_1(x)$ is the minimal polynomial for $\bar{T}$ over F and $p(x)$ is that for T, then prove that $p_1(x)$ divides $p(x)$.
b) If V is a n-dimensional vector space over F and if $T \in A_F(V)$ has all its characteristics roots in F, then prove that T satisfies a polynomial of degree n over F.
c) If $T \in A_F(V)$ is nilpotent then show that $\alpha_0 + \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_m T^m$ is invertible when $\alpha_0 \neq 0$ where $\alpha_i \in F$. **(5+5+4)**

5. a) Define nilpotent transformation. Prove that two nilpotent transformations are similar if and only if they have same invariants.
b) If $T \in A_F(V)$ and $p(x) \in F[x]$ is the minimal polynomial for T over F. Suppose that $p(x) = q_1(x)^{\ell_1} q_2(x)^{\ell_2} \cdots q_k(x)^{\ell_k}$ where the $q_i(x)$ are distinct irreducible polynomials in $F[x]$ and $\ell_1, \ell_2, \cdots, \ell_k$ are positive integers. If $V_i = \{v \in V : v q_i(T)^{\ell_i} = 0\}$ and T_i is a linear transformation induced by T on V_i for $i = 1,2,\cdots,k$, then prove that for each $i = 1,2,\cdots,k$, $V_i \neq 0$ and $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$, the minimal polynomial of T_i is $q_i(x)^{\ell_i}$. **(7+7)**

6. a) State and prove Cauchy Schwartz inequality in an inner product space.
b) Define an orthogonal and orthonormal set of vectors with examples. If A is a symmetric matrix then prove that any two eigen vectors from different eigen spaces are orthogonal.
c) Orthodiagonalize the following matrix $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ **(5+4+5)**

7. a) Classify the following quadratic forms.
(i) $Q(x) = 3x_1^2 + 2x_2^2 + x_3^3 + 4x_1x_2 + 4x_2x_3$
(ii) $Q(x) = 3x_1^2 + 6x_2^2 - 4x_1x_2$
(iii) $Q(x) = 3x_1^2 + 3x_2^2 + 3x_3^3 + 2x_1x_2 + 2x_1x_2 - 2x_2x_3$
(iv) $Q(x) = x_1^2 + x_3^2 + 2x_1x_2 + 2x_2x_3$.

b) Let $Q(x) = 3x_1^2 + 3x_2^2 + 4x_3^3 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3$. Find the maximum value of the quadratic form $Q(x)$ subject to $x^T x = 1$ and also find a unit vector at which this value is attained.
c) Find the singular value decomposition of the matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & 2 \end{bmatrix}$. **(4+4+6)**

8. a) For any n dimensional vector space over F, any basis β for V, prove that Ψ_β is a vector space isomorphism from $B(V)$ onto $M_n(F)$.
b) Define rank and signature of a real quadratic form. Show that two real symmetric matrices are congruent if and only if they have the same rank and signature.
c) Find the rank and signature of $x_1^2 + 4x_2^2 - 3x_3^3$. **(5+7+2)**